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# Seepage in a Saturated-Stratified Aquifer with Recharge[1]

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## ABSTRACT

**An analytical solution for the flow of water through a large-scale laboratory aquifer consisting of three soil layers is presented. It is assumed that each layer is saturated, that the soil is homogeneous but may be anisotropic and that the flow of water in each layer is two-dimensional; that is, the flow at the inflow and outflow boundary is uniform over the entire aquifer width. The solution includes a flux boundary condition at the surface which allows recharge to, or a loss from, the upper aquifer. The solution is presented in terms of the hydraulic head and the stream function. The features of the solution are illustrated with several examples.**

***Additional Index Words:*** **Saturated flow, Stratified aquifer, Analytical solution, Anisotropy, Approximate solutions.**


IN RECENT YEARS there has been considerable interest in describing the heterogeneity of soil and aquifer materials. It has been generally recognized that heterogeneities can have a considerable effect on the movement of water and the substances contained in the water, such as pollutants. Warrick and Nielsen (1980) and Jury (1985) compiled the results from recent field studies and have shown that the coefficient of variation for parameters such as saturated hydraulic conductivity, apparent diffusion coefficient, pore water or solute velocity, etc. can exceed 100%. The major difficulties to date have been in the development of methods which can accurately characterize the spatial heterogeneity in three dimensions and in coupling these techniques to flow and transport models.

Due to the difficulties in fully describing the spatial heterogeneity commonly encountered in the field, laboratory column experiments using homogeneous soils are often used to test theoretical conceptualizations about a given phenomenon. Generally, experiments such as these suffer from the disadvantage that the results are not directly applicable to field problems because of the limiting homogeneity assumption. This has motivated other researchers to use relatively undisturbed samples of soil material for their experiments (Smith et al., 1985; White, 1985; White et al., 1986) which brings a laboratory study closer to what really occurs in a field situation. A disadvantage of this approach is that the microscale heterogeneities are still difficult to describe and the sample is probably not representative of the heterogeneities found at larger scales of observation.

Because of these problems, a current research effort is underway which combines a known heterogeneous pattern in a large-scale laboratory aquifer to determine the effects of large-scale heterogeneous features on the distribution of contaminants in space and time. An advantage of this approach is that the heterogeneities (i.e. soil layering) are known and can be completely described. This makes it possible to model the flow of water, and hence, the interstitial velocity in the stratified aquifer using analytical techniques. This solution can then be incorporated into the solute transport equation to describe the movement of contaminants in the aquifer. Also, since large-scale layering is commonly found in aquifers and near-surface soils, the results should be useful in extrapolation to field scales. A disadvantage is that the simplification of a homogeneous (but possible anisotropic) layer is probably not what commonly occurs in the field.

The effects of layering on the flow of water in the subsurface has been investigated for many years. Muskat (1946) investigated the pressure distribution in a fractured limestone consisting of two layers and derived an analytical solution for a finite region. Freeze and Witherspoon (1966) investigated the regional flow of water in a layered aquifer using numerical and analytical solutions to the flow problem. Selim et al. (1975) and Selim (1987) derived analytical solutions for the flow of water through layered soil with a sloping surface; the latter paper included the effects of anisotropy.

Muskat (1946) and Kirkham and Affleck (1977) investigated the flow pattern in a radial flow field with two concentric zones of permeability surrounding a well. The travel times for the radial flow field were determined by Kirkham and Affleck (1977).

Although the aforementioned literature has investigated the effects of stratifications on the flow of water, none of the solutions presented satisfy the boundary conditions required to describe the flow pattern in the laboratory-scale aquifer. Therefore it is necessary to develop the equations and solutions for a finite system with the appropriate boundary conditions (i.e. boundary conditions that could be constructed in the laboratory).

The first phase of this research project is to develop an analytical solution for the flow of water through the soil layers of the aquifer. The next phase includes using the analytical solution to aid in the design and operation of the stratified aquifer system. The specifics of the design considerations are reported by Beck et al. (1987) and will not be reiterated here. Ultimately the solution contained herein can be used to determine the pore-water velocity as an input to the solute transport equation in an effort to describe the fate and transport of contaminants in a stratified aquifer system.

## THEORY

Shown in Fig. 1 is a saturated-stratified aquifer consisting of three layers. For each layer, the medium is assumed to

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be homogeneous but may be anisotropic. For the purposes of the discussion contained herein, the upper and lower layers are considered to be aquitards and the middle layer an aquifer. The flow of water in each region is assumed to be two-dimensional.

The differential equations which describe the flow of water in the stratified-aquifer system are

$$K_{ix}\frac{\partial^2 H_i}{\partial x^2} + K_{iz}\frac{\partial^2 H_i}{\partial z^2} = 0; \qquad i = 1,2,3 \qquad [1]$$

where $i$ = 1, 2 and 3 for regions I, II, and III, respectively; $H_i$ is the hydraulic head; ($L$), $K_{ix}$, and $K_{iz}$ are the saturated hydraulic conductivities ($L/T$) for the $x$- and $z$-directions, respectively, for region $i$.

The boundary conditions appropriate for each region shown in Fig. 1 are

*Region I and Boundary Between Regions I and II.*

$$\frac{\partial H_1(0,z)}{\partial x} = \frac{\partial H_1(L,z)}{\partial x} = 0;$$

$$\frac{\partial H_1(x,F)}{\partial z} = -q_e/K_{1z} \qquad [2a]$$

$$K_{1z}\frac{\partial H_1(x,a)}{\partial z} = K_{2z}\frac{\partial H_2(x,a)}{\partial z};$$

$$H_1(x,a) = H_2(x,a) \qquad [2b]$$

*Region II*

$$q_o = -K_{2x}\frac{\partial H_2(0,z)}{\partial x}; \quad q_L = -K_{2x}\frac{\partial H_2(L,z)}{\partial x} \qquad [3]$$

*Region III and Boundary Between Regions II and III*

$$\frac{\partial H_3(0,z)}{\partial x} = \frac{\partial H_3(L,z)}{\partial x} = \frac{\partial H_3(x,-G)}{\partial z} = 0 \qquad [4a]$$

$$H_3(x,-a) = H_2(x,-a)\,. \qquad [4b]$$

## Solution for the Stream Function

To solve Eq. [1] subject to Eq. [2] through [4] it is easier to rewrite the problem in terms of the stream function, $\psi$. This is due to the difficulty in determining the functional relationship for $H_2(x,z)$ that gives the appropriate values along the boundaries between regions I & II and II & III and that satisfy the boundary conditions in Eq. [3].

Using the Cauchy-Riemann equations (Kirkham and Powers, 1971; Haberman, 1983; Kreyszig, 1967) for an anisotropic system (Bear, 1972)

$$K_{ix}\frac{\partial H_i}{\partial x} = \frac{\partial \psi_i}{\partial z}; \qquad K_{iz}\frac{\partial H_i}{\partial z} = \frac{-\partial \psi_i}{\partial x} \qquad [5]$$

allows Eq. [1] to be rewritten as

$$\frac{\partial^2 \psi_i}{\partial x^2} + \alpha_i^2\frac{\partial^2 \psi_i}{\partial z^2} = 0 \qquad [6]$$

where $\alpha_i^2$ is the anisotropy ratio and is defined as

$$\alpha_i^2 = K_{iz}/K_{ix}\,. \qquad [7]$$

The appropriate boundary conditions for the stream function at the external boundaries of the stratified aquifer are

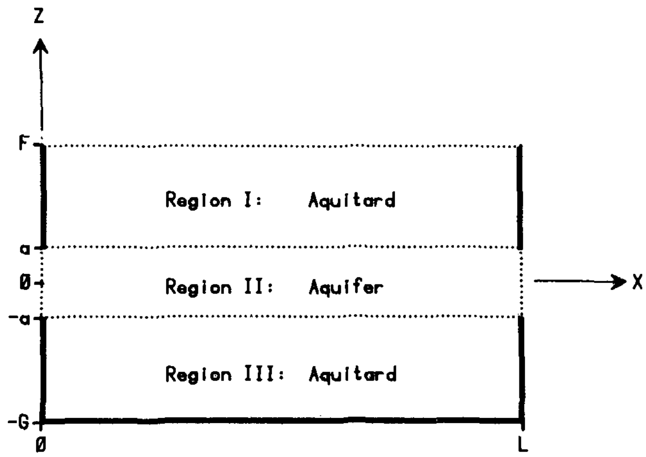


**Fig. 1. Schematic diagram of the stratified-aquifer system where the bold and dotted lines indicate no flow and prescribed flow boundaries, respectively.**

found by defining the external boundary for region III as $\psi$ = 0 and using mass conservation and the definitions

$$\psi_2(0,z) = -\int_{-a}^{z} q_o dz; \quad \psi_2(L,z) = -\int_{-a}^{z} q_L dz;$$

$$\psi_1(x,F) = \int_0^x q_e dx \qquad [8]$$

to determine the boundary values for the stream function in regions I and II, which are:

*Region I*

$$\psi_1(0,z) = -2aq_o; \quad \psi_1(x,F) = -2aq_o + q_e x;$$
$$\psi_1(L,z) = -2aq_o + q_e L \qquad [9]$$

*Region II*

$$\psi_2(0,z) = -(z + a)q_o;$$
$$\psi_2(L,z) = -(z + a)[q_o - q_e L/2a] \qquad [10]$$

*Region III*

$$\psi_3(0,z) = \psi_3(L,z) = \psi_3(x,-G) = 0\,. \qquad [11]$$

At the internal boundaries between regions I and II, Eq. [2b] and the Cauchy-Riemann equations imply that, in terms of the stream function, the condition that the fluxes across the boundary are equal is the same as

$$\frac{\partial \psi_1(x,a)}{\partial x} = \frac{\partial \psi_2(x,a)}{\partial x} \qquad [12]$$

and the condition that the hydraulic heads across the boundary are equal is the same as

$$(1/K_{1x})\int \partial\psi_1/\partial z\, dx + f_1(z)$$
$$= (1/K_{2x})\int \partial\psi_2/\partial z\, dx + g_1(z)\,. \qquad [13]$$

For the boundary between regions II and III, the flux and head conditions, respectively, are the same as

$$\frac{\partial \psi_2(x,-a)}{\partial x} = \frac{\partial \psi_3(x,-a)}{\partial x}; \qquad [14]$$

$$(1/K_{2x})\int \partial\psi_2/\partial z\, dx + f_2(z)$$
$$= (1/K_{3x})\int \partial\psi_3/\partial z\, dx + g_2(z)\,. \qquad [15]$$

Using the separation of variables technique (Churchill and Brown, 1978; Haberman, 1983; Wylie and Barrett, 1982), the solution to Eq. [6] subject to Eq. [9] through [13] can be written in terms of a Fourier series

$$\psi_1(x,z) = -2aq_o + q_c x + \sum_{n=1}^{\infty} A_n \sin(kx)\sinh[k(F - z)/\alpha_1] \quad [16]$$

$$\psi_2(x,z) = -(a + z)(q_o - q_c x/2a) + \sum_{n=1}^{\infty} \sin(kx)[V_n \sinh(kz/\alpha_2) + W_n \cosh(kz/\alpha_2)] \quad [17]$$

$$\psi_3(x,z) = \sum_{n=1}^{\infty} C_n \sin(kx)\sinh[k(G + z)/\alpha_3] \quad [18]$$

where $k = n\pi/L$.

Since $\psi_1$, $\psi_2$, and $\psi_3$ are written in a form that satisfies the external boundary conditions of the stratified aquifer system, the next step in arriving at the solution for the stream function and subsequently the hydraulic head is to determine the coefficients: $A_n$, $V_n$, $W_n$, and $C_n$.

Equating the fluxes at the boundary between region I and II, using Eq. [12], grouping terms under a common series and manipulating the result gives a relationship between $A_n$, $V_n$, and $W_n$

$$A_n = \frac{V_n \sinh(ka/\alpha_2) + W_n \cosh(ka/\alpha_2)}{\sinh[k(F - a)/\alpha_1]} \quad [19]$$

Equating the hydraulic head at the boundary using Eq. [13], accumulating the terms under the series and integrating both sides of the equation using the principles of orthogonality gives one of two required relationships between $V_n$ and $W_n$

$$V_n\xi_1 + W_n\xi_2 = \lambda_n \sqrt{K_{2x} K_{2z}} = \bar{\lambda}_n \quad [20]$$

where

$$\bar{\lambda}_n = 2\alpha_2\{q_o[1 - (-1)^n] + (-1)^n q_c L/(2a)\}/(Lk^2) \quad [21]$$

and the $\xi_i$'s are various combinations of the hyperbolic sines and cosines shown in Eq. [16] through [18]. For brevity, explicit expressions for the $\xi_i$'s are not provided since they are not part of the final solution for the hydraulic head or stream function given below.

Repeating this procedure at the boundary between regions II and III gives a relationship between $C_n$, $V_n$, and $W_n$

$$C_n = \frac{-V_n \sinh(ka/\alpha_2) + W_n \cosh(ka/\alpha_2)}{\sinh[k(G - a)/\alpha_3]} \quad [22]$$

and a second relationship between $V_n$ and $W_n$

$$V_n \xi_3 - W_n \xi_4 = \bar{\lambda}_n . \quad [23]$$

Multiplying Eq. [20] by $\xi_3$ and Eq. [23] by $\xi_1$, subtracting and solving gives an explicit expression for $W_n$

$$W_n = \frac{\bar{\lambda}_n[P_1 - P_3]P_2}{\cosh(ka/\alpha_2)\,\zeta_n} \quad [24]$$

where

$$\zeta_n = (P_1 + P_3)[1 + (P_2)^2] + 2P_2(P_1P_3 + 1) \quad [25]$$

and

$$P_1 = [K_{1x}K_{1z}/(K_{2x}K_{2z})]^{1/2}\tanh[k(F - a)/\alpha_1]$$

$$P_2 = \tanh(ka/\alpha_2)$$

$$P_3 = [K_{3x}K_{3z}/(K_{2x}K_{2z})]^{1/2}\tanh[k(G - a)/\alpha_3] . \quad [26]$$

Substituting $W_n$ into either Eq. [20] or Eq. [23] provides a means for determining $V_n$

$$V_n = \frac{\bar{\lambda}_n [P_1 + P_3 + 2(P_1P_2P_3)]}{\cosh(ka/\alpha_2)\,\zeta_n} . \quad [27]$$

The solution to the stream function in the stratified-aquifer system is found by using Eq. [16] through [19], [21], [22] and [24] through [27].

## Solution for the Hydraulic Head

The equations which describe the hydraulic head in the stratified aquifer can be found by using the Cauchy-Riemann equations given in Eq. [5]. Integrating both relationships in Eq. [5] and finding the arbitrary functions gives equations for the hydraulic head in each region

$$H_1(x,z) = A_o - q_c z/K_{1z} + (\alpha_1/K_{1z}) \sum_{n=1}^{\infty} A_n \cos(kx)\cosh[k(F - z)/\alpha_1] \quad [28]$$

$$H_2(x,z) = B_o - q_o x/K_{2x} - q_c z/(2K_{2z}) \quad [29]$$
$$+ q_c[x^2/K_{2x} - z^2/K_{2z}]/(4a) - (\alpha_2/K_{2z}) \times \sum_{n=1}^{\infty} \cos(kx)[V_n \cosh(kz/\alpha_2) + W_n \sinh(kz/\alpha_2)]$$

$$H_3(x,z) = C_o - (\alpha_3/K_{3z}) \times \sum_{n=1}^{\infty} C_n \cos(kx)\cosh[k(G+z)/\alpha_3] . \quad [30]$$

Since only the flux is specified at the $x = 0$ boundary in region II (i.e. the outflow boundary value results from mass conservation), another boundary condition is required to fully specify the constants $B_o$, and hence, $A_o$ and $C_o$. Given the boundary condition, $H_2(0, 0) = H_o$, the constants are

$$B_o = H_o + (\alpha_2/K_{2z}) \sum_{n=1}^{\infty} V_n \quad [31]$$

$$A_o = B_o + q_c a[1/K_{1z} - 0.75/K_{2z}] - q_o L/(2K_{2x}) + q_c L^2/(12aK_{2x}) \quad [32]$$

$$C_o = B_o + q_c a/(4K_{2z}) - q_o L/(2K_{2x}) + q_c L^2/(12aK_{2x}) \quad [33]$$

which completes the derivation and provides a unique solution for both the hydraulic head and stream function.

## Inflow and Outflow Regions

The flow pattern in the saturated stratified aquifer consists of an inflow region where the flow is diverging from the entrance boundary, a middle region where the flow lines are relatively straight and conditions are approximately constant, and an outflow region where the flow is converging toward the exit boundary.

The extent of the inflow and outflow regions in the $x$-direction can be approximated by determining the position in each layer where the slope of the stream function is approximately the same as the slope at $x = L/2$,

$$|\theta(x, z_m) - \theta(L/2, z_m)| = \left|\tan^{-1}\left[\frac{\partial\psi(x, z_m)/\partial x}{-\partial\psi(x, z_m)/\partial z}\right] - \theta(L/2, z_m)\right| \leq \epsilon \quad [34]$$

where $\theta(x, z_m)$ is the angle of the stream function from the horizontal at the point $(x, z_m)$ and is given as the inverse tangent of the ratio of the $z$- and $x$-direction velocities in the middle part of Eq. [34], $z_m$ is some arbitrary point in the layer taken herein as the midpoint of the layer and $\epsilon$ is an arbitrarily small error criterion, which is the maximum allowed angular deviation.

In the lower aquifer $\theta(L/2, z_m) \simeq 0$ which may simplify Eq. [34]. The angle $\theta(L/2, z_m)$ is also found using the inverse tangent but once it is found remains constant for a given layer.

## Approximate Solutions

In the middle region of the aquifer, where the $x$- and $z$-gradients of the hydraulic head and stream function, respectively, are approximately constant, it is possible to obtain approximate solutions. If the approximation

$$\frac{\partial\psi_i(x,z)}{\partial z} \simeq \frac{\partial\psi_i(x,z_m)}{\partial z} \simeq b_i\frac{\partial\psi_2(x,0)}{\partial z} = b_iA(x) \quad [35]$$

is adopted, where $z_m$ is an arbitrary point in the aquifer and $b_i$ and $A(x)$ are to be determined, then simple approximate solutions for the hydraulic head and stream function in the middle region can be found by integrating Eq. [35] and using the boundary conditions in Eq. [2] to [4] and [9] to [11]. The approximate solutions for the stream function are

$$\psi_1(x,z) = -2aq_o + q_ex - A(x)[K_{1x}/K_{2x}](F - z) \quad [36]$$

$$\psi_2(x,z) = \psi_1(x,a) - A(x)[a - z] \quad [37]$$

$$\psi_3(x,z) = (K_{3x}/K_{2x})A(x)[G + z] \quad [38]$$

where $b_i = K_{ix}/K_{2x}$ and is found by using Eq. [13] and [15], and $A(x)$ is an approximation to the gradient of the stream function, $\partial\psi_2(x,0)/\partial z$, and is taken to be independent of $z$

$$A(x) = -q_o + q_ex/(2a) + \sum_{n=1}^{\infty} V_n\, k \sin(kx)/\alpha_2 . \quad [39]$$

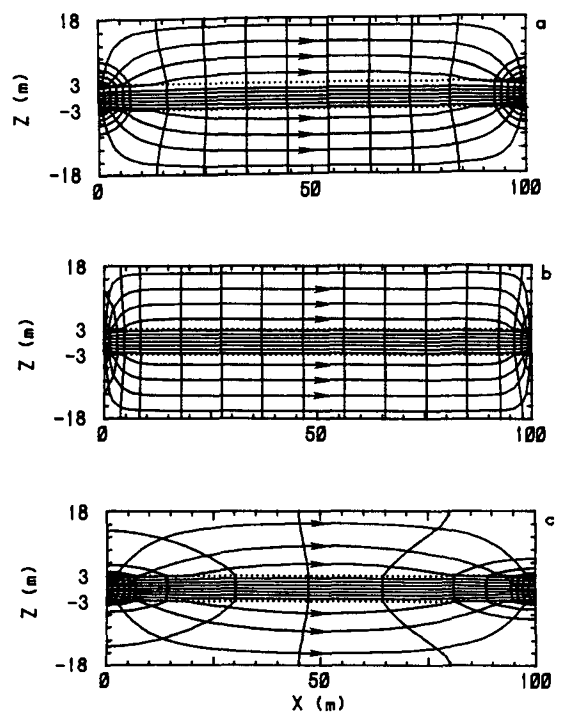


**Fig. 2. Hydraulic head and stream function for a no-flow boundary condition at $z = F$. In a, b and c, respectively, $\alpha_i^2$ = 1.0, 10.0, and 0.1. The contour levels are given in Tables 1 and 2.**

By making a further approximation that the value of the series in Eq. [39] at $x$ is approximately the same as its value at $L/2$, $A(x)$ can be simplified

$$A(x) \simeq -q_o + \sum_{n=1}^{\infty} V_n\, k \sin(kL/2)/\alpha_2 + q_ex/(2a) = -q_c + q_ex/(2a) \quad [40]$$

where $q_c$ is a constant. The value for the series in Eq. [40], (i.e. the term $q_c$) need only be determined once for each problem.

The relationships for the hydraulic head can be found from the stream function by using the Cauchy-Riemann equations and are

$$H_1(x,z) \simeq H_2(L/2,0) + q_e\left[\frac{1}{K_{1x}} - \frac{F}{2aK_{2x}}\right](a - z)/\alpha_1^2 + \frac{q_c}{K_{2x}}(L/2 - x) - \frac{q_e}{4aK_{2x}}[(L/2)^2 - x^2 - (a^2 - z^2)/\alpha_1^2] \quad [41]$$

$$H_2(x,z) \simeq H_2(L/2,0) - q_e\left[\frac{1}{2} - \frac{(F-a)K_{1x}}{2aK_{2x}}\right]z/K_{2z} + \frac{q_c}{K_{2x}}(L/2 - x) - \frac{q_e}{4aK_{2x}}[(L/2)^2 - x^2 + z^2/\alpha_2^2] \quad [42]$$

$$H_3(x,z) \simeq H_2(L/2,0) - q_e\frac{G}{2aK_{2x}}(a + z)/\alpha_3^2 + \frac{q_c}{K_{2x}}(L/2 - x) - \frac{q_e}{4aK_{2x}}[(L/2)^2 - x^2 - (a^2 - z^2)/\alpha_3^2] \quad [43]$$

## EXAMPLES

Several examples which illustrate the analytical solution contained herein show the versatility of the solution in describing physical systems of various configurations. The figures were constructed without vertical exaggeration to show that the potential and streamlines are perpendicular when the anisotropy ratio is unity. For all the examples, it is assumed that $H(0,0)$ = 5 m and $q_o$ = 0.785 m/d. Unless otherwise noted, the ratios of the hydraulic conductivities $K_{1x}/K_{2x}$ and $K_{3x}/K_{2x}$ are 0.1 and the anisotropy ratio for each layer is the same. In the aquifer, $K_{2x}$ = 10 m/d for $\alpha_i^2$ = 1 or 0.1 and $K_{2x}$ = 1 m/d when $\alpha_i^2$ = 10.0. The aquifer thickness is taken to be 6.0 m and the total thickness of the stratified aquifer system is 36.0 m. To calculate a value for the hydraulic head and stream function, 100 terms were used in the series in Eq. [16] through [18] and [28] through [30]. For the problem described in Fig. 2, however, using 5 and 10 terms would have produced results which were in error by less than about 1.0% and 0.2%, respectively, at the nine points described by the coordinates $x/L$ = 0.01, 0.05, 0.20 and $z = (f - a)/2$, 0, $(G - a)/2$. The figures were created by calculating the values for the hydraulic head and stream function at 1887 points on a uniform grid system ($\Delta x$ = 2 m, $\Delta z$ = 1 m) and determining the contours using the routine described

**Table 1. Contour interval definition for the hydraulic head in Fig. 2 through 6.†**

| Figure | Contour level description for hydraulic head |
|---|---|
| 2a | [4.9, 4.5, 0.1]; [4.0, 0.5, 0.5]; [−0.1, −0.5, 0.1] |
| 2b | [4.0, 2.0, 1.0]; [0.0, −40, 5.0]; −44, [−46, −48, 1] |
| 2c | [4.9, 4.5, 0.1]; [4.0, 0.0, 1.0]; −0.5, [−0.9, −1.2, 0.1] |
| 3a | [4.9, 4.5, 0.1]; 4.3, [4.0, 0.5, 0.5]; [0.4, 0.1, 0.1] |
| 3b | [4.0, 2.0, 1.0]; [0.0, −35, 5.0]; [−40, −42, 1.0]; −42.5 |
| 3c | [4.9, 4.5, 0.1]; [4.0, 0.5, 0.5]; [0.1, −0.6, 0.1] |
| 4a | [4.9, 4.5, 0.1]; [4.0, −0.5, 0.5]; [−0.8, −1.1, 0.1] |
| 4b | [4.0, −4.0, 2.0]; [−10, −40, 5.0]; [46, 52, 2.0] |
| 4c | [4.8, 4.5, 0.1]; [4.0, −1.0, 0.5]; [−1.5, −1.8, 0.1] |
| 5a | [4.9, 4.5, 0.1]; [4.0, 1.0, 0.5]; [0.7, 0.4, 0.1] |
| 5b | [4.0, 2.0, 1.0]; [0.0, −35, 5.0]; [−37, −39, 1.0] |
| 5c | [4.9, 4.5, 0.1]; [4.0, 0.0, 0.5]; [−0.1, −0.4, 0.1] |
| 6a | [4.9, 4.5, 0.1]; [4.0, 1.0, 0.5]; 0.7, [0.4, 0.0, 0.1] |
| 6b | [4.0, 2.0, 1.0]; [0.0, −35, 5.0]; −38, [−40, −42, 1.0] |
| 6c | [4.9, 4.5, 0.1]; [4.0, 0.0, 0.5]; [−0.5, −0.9, 0.1] |

† For example, in Fig. 2c the contour levels are: 4.9, 4.8, 4.7, 4.6, 4.5, 4.0, 3.0, 2.0, 1.0, 0.0, −0.5, −0.9, −1.0, −1.1 and −1.2.

**Table 2. Contour levels definition for the stream function in Fig. 2 through 6.†**

| | Contour level descriptions for the stream function | | |
|---|---|---|---|
| | Region | | |
| Figure | I | II | III |
| 2a | [0.98, 0.86, 0.04] | [0.7, 0.2, 0.1] | [0.14, 0.02, 0.04] |
| 2b | [0.98, 0.86, 0.04] | [0.8, 0.2, 0.1] | [0.14, 0.02, 0.04] |
| 2c | [0.98, 0.90, 0.04] | [0.8, 0.2, 0.1] | [0.10, 0.02, 0.04] |
| 3a | [0.98, 0.74, 0.04] | [0.7, 0.2, 0.1] | [0.14, 0.02, 0.04] |
| 3b | [0.98, 0.70, 0.04] | [0.6, 0.2, 0.1] | [0.14, 0.02, 0.04] |
| 3c | [0.98, 0.78, 0.04] | [0.7, 0.2, 0.1] | [0.10, 0.02, 0.04] |
| 4a | [1.15, 0.90, 0.05] | [0.8, 0.2, 0.1] | [0.14, 0.02, 0.04] |
| 4b | [1.15, 0.90, 0.05] | [0.8, 0.2, 0.1] | [0.14, 0.02, 0.04] |
| 4c | [1.15, 1.00, 0.05] | [0.9, 0.2, 0.1] | [0.10, 0.02, 0.04] |
| 5a | [0.98, 0.70, 0.04] | [0.6, 0.1, 0.1] | [0.06, 0.02, 0.04] |
| 5b | [0.98, 0.62, 0.04] | [0.5, 0.2, 0.1] | [0.06, 0.02, 0.04] |
| 5c | [0.98, 0.70, 0.04] | [0.6, 0.2, 0.1], 0.06 | 0.02 |
| 6a | [0.98, 0.66, 0.04] | [0.6, 0.1, 0.1] | 0.02 |
| 6b | [0.98, 0.70, 0.04] | [0.6, 0.1, 0.1] | 0.02 |
| 6c | [0.98, 0.78, 0.04] | [0.7, 0.1, 0.1] | 0.02 |

† For example the contour levels in Fig. 5c are: Region I: 0.98, 0.94, 0.90, 0.86, 0.82, 0.78, 0.74, 0.70; Region II: 0.6, 0.5, 0.4, 0.3, 0.2, 0.06; and for Region III: 0.02.

by Yates (1987). For brevity in listing the contour levels, the notation $[a,b,c]$ is adopted where $a$ and $b$ are the beginning and ending values for a sequence of contour levels and $c$ is the difference between adjacent contour levels. The contour levels for each figure are given in Tables 1 and 2, respectively, for the hydraulic head and stream functions.

Shown in Fig. 2 are the hydraulic head and streamlines when a no-flow boundary condition exists at the upper surface, $z = F$. For Fig. 2a, 2b and 2c, respectively, the anisotropy ratios are 1.0, 10.0, and 0.1.

As expected when $\alpha_i^2 = 1$, the potential and streamlines are perpendicular. By comparing Fig. 2a and 2b, the effects of increasing the anisotropy ratio can be determined. For this example, the streamlines in the central part of the aquifer system are the same whether $\alpha_i^2 = 1$ or 10. Near the entrance and exit, however, the streamlines when $\alpha_i^2 = 10$ have a steeper slope which follow the side of the aquifer system more closely. The hydraulic head lines for $\alpha_i^2 = 1$ and $\alpha_i^2 = 10$ are very different. When $\alpha_i^2 = 10$, the $x$-direction hydraulic conductivity is 10 times smaller when compared to $\alpha_i^2 = 1$, therefore, the hydraulic gradient must be increased if the same flux of water is to move through the stratified aquifer system. This causes the large negative values for the hydraulic head at the exit boundary when $\alpha_i^2 = 10.0$.

When the anisotropy ratio is reduced from a value of 1.0 to 0.1, the positioning of the potential and streamlines differ markedly. For $\alpha_i^2 = 0.1$, the streamlines have a shallower slope near the entrance and exit and the maximal elevation for a given streamline in Region I (and minimal elevation in Region III) is closer to the aquifer compared to the $\alpha_i^2 = 1$ case. The positioning of the potential lines for the $\alpha_i^2 = 0.1$ case is similar to the $\alpha_i^2 = 1$ case since the $x$-direction hydraulic conductivities for each is the same. There is a slight increase in the hydraulic gradient between the entrance and exit boundaries due to a small increase in resistance caused by the higher $z$-direction conductivities for the $\alpha_i^2 = 0.1$ case. These same observations are found in the remaining figures.

The effect of anisotropy on the fraction of water which is transported from the entrance to the exit boundaries entirely within the aquifer, $f_2$, can be determined by using

$$f_2 = \bar{\psi}_2(50,a) - \bar{\psi}_2(50, -a) \qquad [44]$$

where $\bar{\psi}_2 = -\psi_2/2aq_o$ is a normalized stream function and $0 < \bar{\psi}_2 < 1$ for this example. For anisotropy ratios of 1.0, 10.0, and 0.1, respectively, $f_2$ is 66.78%, 66.67%, and 73.72%. The higher value of $f_2$ for $\alpha_i^2 = 0.1$ demonstrates that the aquifer acts as a preferential flow path when $\alpha_i^2 \leq 1.0$. Also, as $\alpha_i^2 \rightarrow \infty$, it can be shown that

$$f_2 \rightarrow \{1 + [(F - a)K_{1x} + (G - a)K_{3x}/2aK_{2x}\}^{-1} \qquad [45]$$

as $K_{2z} \rightarrow \infty$, which means that the stratified aquifer system is perfectly mixed in the $z$-direction. (The alternative, $K_{2x} \rightarrow 0$, is not useful based on physical grounds since as $K_{2x} \rightarrow 0$, flow will result in the $x$-direction only for an infinite gradient which is physically unrealizable). Therefore, the flow in the aquifer is the ratio of the aquifer thickness to the total thickness weighted by the appropriate conductivities. For this example, as $\alpha_i^2 \rightarrow \infty$, $f_2 \rightarrow 2/3$.

The effects due to a loss of water at the upper surface is shown in Fig. 3, where in 3a, 3b, and 3c, respectively, the anisotropy ratio is 1.0, 10.0 and 0.1. The flux through the upper surface is $q_e = 0.01$ m/d which represents about 1.25% of the flux into the aquifer at the entrance boundary. Comparing Fig. 2 and 3 shows the effects of mass loss on the potential and streamlines.

Comparing Fig. 3a, 3b, and 3a, 3c shows a similar behavior of the potential and streamlines to that found in Fig. 2. The point where the streamline connects to the upper surface is not affected by increasing the anisotropy ratio, although for Fig. 3a and 3b the slope of the streamlines near the aquifer entrance and exit does depend on the anisotropy ratio.

When a surface flux condition exists, a stagnation point may be present. The position of the stagnation point can be found by setting either $x = L$ (for $q_e > 0$) or $x = 0$ (for $q_e < 0$) so that the flux in the $x$-direction is zero and solving for the $z$ which satisfies

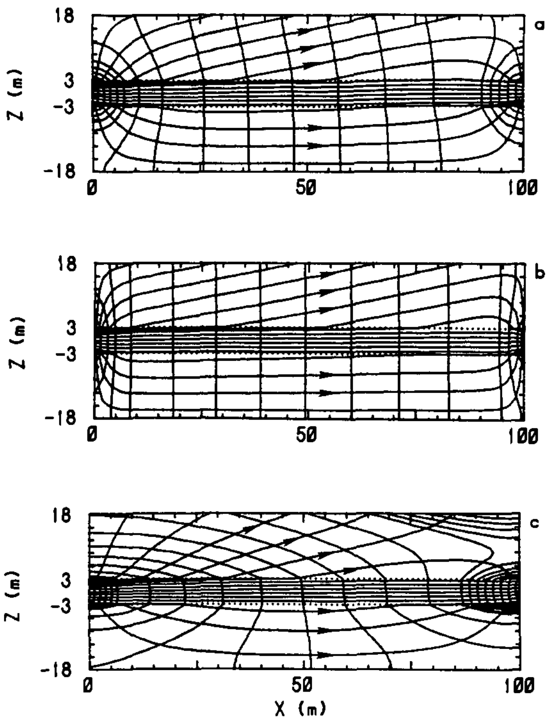


Fig. 3. Hydraulic head and stream function when the flow out of the upper surface is $q_e = 0.01$ m/d. In a, b, and c, respectively, $\alpha_i^2 = 1.0$, 10.0, and 0.1. The contour levels are given in Tables 1 and 2.

$$q_e + \sum_{n=1}^{\infty} A_n (-1)^n k \sinh[k(F - z)/\alpha_1] = 0; \; q_e > 0$$

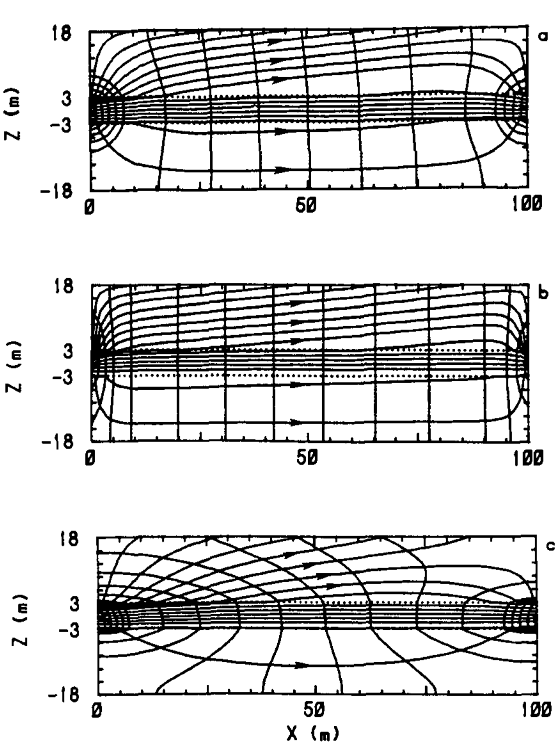


Fig. 5. Hydraulic head and stream function when $q_e = 0.01$ m/d and the ratios $K_{1x}/K_{2x}$ and $K_{3x}/K_{2x}$ are 0.2 and 0.05, respectively. In a, b, and c, respectively, $\alpha_i^2 = 1.0$, 10.0, and 0.1. The contour levels are given in Tables 1 and 2.

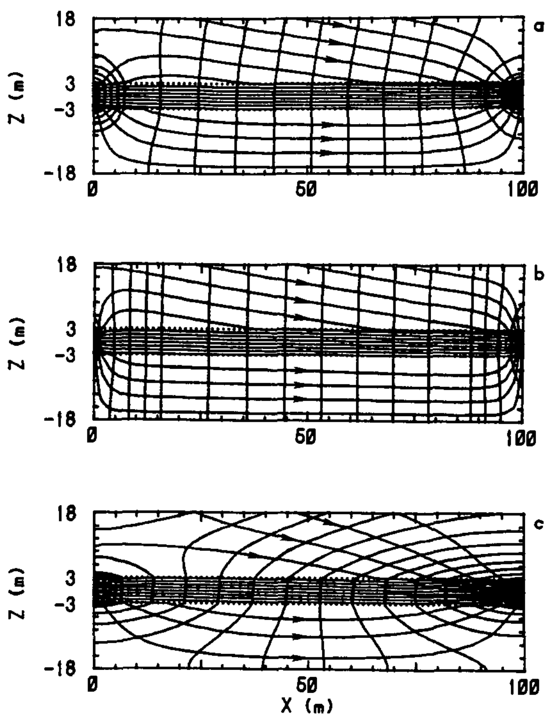


Fig. 4. Hydraulic head and stream function when the flow into the stratified aquifer is $q_e = -0.01$ m/d. In a, b, and c, respectively, $\alpha_i^2 = 1.0$, 10.0, and 0.1. The contour levels are given in Tables 1 and 2.

$$q_e + \sum_{n=1}^{\infty} A_n \, k \sinh[k(F - z)/\alpha_1] = 0; \; q_e < 0 \quad [46]$$

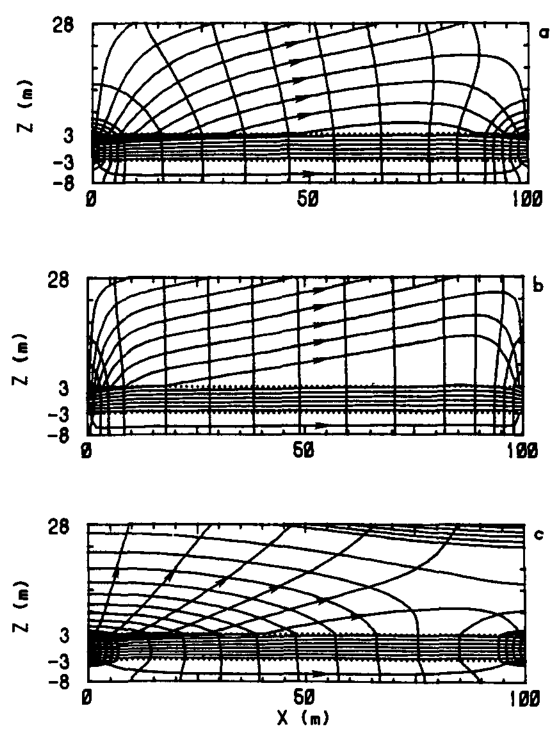


Fig. 6. Hydraulic head and stream function when $q_e = 0.01$ m/d and the thickness of regions I, II, and III, respectively, are 25 m, 6 m, and 5 m. The ratios $K_{1x}/K_{2x} = 0.1 = K_{3x}/K_{2x}$. In a, b, and c, respectively, $\alpha_i^2 = 1.0$, 10.0, and 0.1. The contour levels are given in Tables 1 and 2.

**Table 3. Comparison between exact and approximate and solutions for the hydraulic head and normalized stream function at selected points.**

| $x$ | $z$ | $H$ | $H_{ap}$ † | $\bar{\psi}$ | $\bar{\psi}_{ap}$ ‡ | $\theta(x,z)$ |
|---|---|---|---|---|---|---|
| | | | $\alpha_i^2 = 1.0$ | | | |
| 20.0 | −15.0 | 3.76 | 3.84 | 0.028 | 0.033 | −0.0427 |
| 20.0 | −9.0 | 3.78 | 3.84 | 0.086 | 0.099 | −0.0979 |
| 20.0 | 0.0 | 3.81 | 3.82 | 0.479 | 0.462 | 0.0096 |
| 20.0 | 9.0 | 3.72 | 3.78 | 0.872 | 0.858 | 0.3008 |
| 20.0 | 15.0 | 3.64 | 3.72 | 0.930 | 0.925 | 0.2627 |
| 30.0 | −15.0 | 3.30 | 3.33 | 0.030 | 0.032 | −0.0070 |
| 30.0 | −9.0 | 3.30 | 3.33 | 0.091 | 0.096 | −0.0137 |
| 30.0 | 0.0 | 3.30 | 3.31 | 0.468 | 0.457 | 0.0101 |
| 30.0 | 9.0 | 3.24 | 3.27 | 0.846 | 0.840 | 0.2180 |
| 30.0 | 15.0 | 3.18 | 3.21 | 0.906 | 0.904 | 0.2153 |
| 40.0 | −15.0 | 2.83 | 2.84 | 0.030 | 0.031 | 0.0031 |
| 40.0 | −9.0 | 2.82 | 2.83 | 0.091 | 0.093 | 0.0113 |
| 40.0 | 0.0 | 2.82 | 2.82 | 0.458 | 0.451 | 0.0104 |
| 40.0 | 9.0 | 2.76 | 2.77 | 0.825 | 0.822 | 0.1961 |
| 40.0 | 15.0 | 2.70 | 2.72 | 0.885 | 0.884 | 0.2052 |
| 50.0 | −15.0 | 2.36 | 2.36 | 0.030 | 0.030 | 0.0069 |
| 50.0 | −9.0 | 2.35 | 2.35 | 0.089 | 0.090 | 0.0208 |
| 50.0 | 0.0 | 2.34 | 2.34 | 0.447 | 0.446 | 0.0107 |
| 50.0 | 9.0 | 2.29 | 2.29 | 0.805 | 0.804 | 0.1913 |
| 50.0 | 15.0 | 2.23 | 2.24 | 0.864 | 0.864 | 0.2052 |
| | | | $\alpha_i^2 = 0.1$ | | | |
| 30.0 | −15.0 | 2.61 | 3.24 | 0.016 | 0.035 | −0.0561 |
| 30.0 | −9.0 | 2.78 | 3.18 | 0.053 | 0.105 | −0.1147 |
| 30.0 | 0.0 | 3.06 | 3.06 | 0.468 | 0.409 | 0.0089 |
| 30.0 | 9.0 | 2.15 | 2.58 | 0.883 | 0.831 | 0.3777 |
| 30.0 | 15.0 | 1.38 | 2.04 | 0.921 | 0.901 | 0.4228 |
| 40.0 | −15.0 | 2.34 | 2.69 | 0.018 | 0.034 | −0.0225 |
| 40.0 | −9.0 | 2.41 | 2.63 | 0.059 | 0.102 | −0.0449 |
| 40.0 | 0.0 | 2.51 | 2.51 | 0.458 | 0.404 | 0.0093 |
| 40.0 | 9.0 | 1.78 | 2.03 | 0.856 | 0.813 | 0.3003 |
| 40.0 | 15.0 | 1.11 | 1.49 | 0.897 | 0.881 | 0.3540 |
| 50.0 | −15.0 | 2.05 | 2.17 | 0.019 | 0.033 | 0.0029 |
| 50.0 | −9.0 | 2.03 | 2.11 | 0.061 | 0.099 | 0.0095 |
| 50.0 | 0.0 | 1.99 | 1.99 | 0.447 | 0.400 | 0.0096 |
| 50.0 | 9.0 | 1.40 | 1.51 | 0.833 | 0.795 | 0.2487 |
| 50.0 | 15.0 | 0.82 | 0.97 | 0.875 | 0.861 | 0.3222 |

† $H_{ap}$ = approximate solution for the hydraulic head.
‡ $\bar{\psi}_{ap}$ = approximate solution for the normalized stream function.

that is, the $z$ where the flux in the $z$-direction is zero. In Fig. 3a, 3b, and 3c, respectively, the stagnation points are located at (100, 14.86 m), (100, 16.97 m) and (100, 9.35 m).

The fraction of flow which passes entirely within the aquifer, $f_2$, for $\alpha_i^2 = 1.0$, 10.0, and 0.1, respectively, is 54.57%, 51.38%, and 62.54%.

Shown in Fig. 4 are the potential and streamlines when recharge to the upper surface is considered. Applying a transformation, $\bar{x} = -x$, to Fig. 3a produces a streamline pattern similar to Fig. 4a (provided that the stream function in Fig. 4 is normalized so that $0 \leq \psi \leq 1$). This can be seen by comparing the 1.15 contour line in Fig. 4a (i.e. a normalized contour level of 0.949) to the 0.94 contour level in Fig. 3a.

When a recharge boundary condition exists at the upper surface, a stagnation point may exist which is located where $q_x$ and $q_z$ equal zero. For Fig. 4a, 4b, and 4c, respectively, the coordinates of the stagnation points are (0, 15.49 m), (0, 17.182 m) and (0, 10.636 m).

The fraction of the water that enters the aquifer at the entrance and passes through the system completely within the aquifer for $\alpha_i^2 = 1.0$, 10.0, and 0.1, respectively, is 69.12, 65.65, and 78.72%.

Figure 5 shows the position of the potential and streamlines for three anisotropy ratios when the conductivity of the upper and lower regions, respectively, are 2/10 and 1/20 of the aquifer. Immediately visible is that more of the flow moves through the upper region compared to the lower region (see Fig. 3). In other respects, the overall behavior is similar to that described in Fig. 3.

The stagnation points for Fig. 5a, 5b, and 5c, respectively, are: (100, 16.28 m), (100, 17.44 m) and (100, 10.38 m) and the fraction of flux passing completely through the aquifer is 51.86, 48.91, and 60.26%, respectively.

Figure 6 shows the potential and streamlines when the upper and lower regions have different thicknesses. The conductivity values for the upper and lower regions are 1/10 that of the aquifer (i.e. $K_{1x} = K_{3x} = 0.1K_{2x}$).

In a similar manner to Fig. 5, more of the flow moves through the upper region compared to the lower due to the greater thickness. The stagnation points for Fig. 6a, 6b and 6c, respectively, are: (100, 22.79 m), (100, 26.26 m) and (100, 11.23 m) and the fraction of flow passing through the stratified aquifer system entirely within the aquifer are: 58.92%, 55.15%, and 67.16%, respectively.

Table 3 provides a comparison of the exact and approximate solutions for the hydraulic head and the normalized stream functions shown in Fig. 3. In general, the approximate solution provides more accurate results for larger $\alpha_i^2$. Therefore, for brevity, only the results for $\alpha_i^2 = 1$ and $\alpha_i^2 = 0.1$ are shown in Table 3.

To use the approximate solution it is first necessary to determine the location of the inflow and outflow regions using Eq. [34] such that the conditions on which the approximate solution were based are approximately satisfied. These results are given in column 7 of Table 3. When $\alpha_i^2 = 1$ and $\alpha_i^2 = 0.1$, respectively, and given a 2.5% error criterion (i.e. $|\epsilon| = 0.025\pi$), the inflow region ends at approximately 22.5 m and 35.3 m. Approximate values for the hydraulic head (column 4) and normalized stream function (column 6) can be obtained from Eq. [36] through [38] and [41] through [43] (see Eq. [44] for the definition of the normalized stream function) using a value for $q_c$ of 0.5519 m/d and 0.6017 m/d, respectively for the $\alpha_i^2 = 1$ and $\alpha_i^2 = 0.1$ cases.

The results in Table 3 show that the assumed constant flow condition in the middle part of the aquifer is reasonably well satisfied when $\alpha_i^2 \simeq 1$. When $\alpha_i^2 \simeq 0.1$, on the other hand, the differences between the exact and approximate solutions are much larger. This is due in part to the relatively nonconstant behavior for the $z$-gradient of the stream function in the middle part of the aquifer which exists under these conditions. This behavior can also be seen by noting the placement of the contours of the hydraulic head and stream function in Fig. 3c. If an accurate approximate solution is required for $\alpha_i^2 \ll 1$, more elaborate approximate solution techniques should be investigated.

## CONCLUSIONS

An analytical solution for the potential and stream functions for the flow of water in a saturated-stratified aquifer consisting of three layers has been derived and illustrated. It has been assumed that each layer is homogeneous and anisotropic and the flow is at steady-state. The solution includes flux boundary conditions at the upper surface and at the entrance and exit boundaries of the middle region.

The solution should prove useful to describe the pore water velocity in a stratified-aquifer system which can be incorporated into the solute transport equation to describe the movements of contaminants contained in the water. The solution can also be used to verify the accuracy of steady-state saturated flow solutions such as finite-elements and/or finite difference solutions.

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