ESTIMATION OF HYDRAULIC CONDUCTIVITY WITHOUT COMPUTING FLUXES

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Field estimates of the hydraulic conductivity (K) have large variances resulting from interpolating and differencing errors, in addition to instrumental and other errors in the observed water contents (8) and pressure heads (h). The resulting total error can easily be larger than the estimated value of K, thus producing poor results when analytical functions are fitted to the data. In contrast to *K*, the slope of *K*, i.e., $dK/d\theta$, *can be* estimated without differencing field data, and hence always produces stable (i.e., positivej estimates. The purpose of this study was to compare hydraulic functions fitted to different combinations of measured field data of *θ, h, K, and* $K' = dK/d\theta$, where K' =z/t during the drainage phase of infiltration-drainage experiments. Five combinations of data sets were used in the fitting process, i.e., ***h-8;*** K-8; K' *-θ*; *h-θ-K*; and h-0-K'. The data sets were for the Norfolk sand, Troup loamy sand Bethany loam, and Muir silt loam soils. The ***h-0-K*** or ***h-8-K'*** data consistently produced good tits to both the ***K(8)*** and ***h(8)*** data sets. While nearly overlapping curves were produced over the range of measured field data, the fitted values of the parameters were not always similar. Likewise, using either ***9-K or 8-K*** observations resulted in similar curves, although the results were less consistent than those obtained with the more complete ***h-0-K*** or ***h-0-K'*** data sets. For the soils studied, the *K* and K' data produced very similar curves indicating that differencing field data can be avoided since neither water fluxes nor gradients in the pressure head are needed to estimate K'.

INTRODUCTION

One of the most popular methods for estimating the unsaturated soil hydraulic properties from field data is the instantaneous profile method of ***Rose et aL*** [1965]. This method requires considerable experimental effort in that water content (*θ*) and pressure head *(h)* data must be obtained at regular intervals in space and time. While the precision of the measured ***h*** and ***θ*** data is limited by the precision of the field instruments, hydraulic conductivity (K) values have additional errors arising from differencing inherently noisy water content and pressure head data in naturally heterogeneous field soils. The resulting field estimates can fluctuate widely about the fitted analytical functions ***[Fluhler et aL, 1976].*** In order to reduce the magnitude of the fluctuations in the instantaneous profile method, the field data can be smoothed prior to differencing ***[Ahuja et al,*** 1980; Libardi ***et al.*** 1980; ***Luxmoore et aL,*** 1981]. Smoothing of the data requires subjective decisions about the type of smoothing algorithm to be selected, and the degree of smoothing to be done. While smoothing effects are minimal on ***θ***, their effects on K estimates are less understood. An alternative to working with differenced data, with or without smoothing, is to use unit gradient (or fixed gradient) models, and to fit $dK/d\theta$ functions directly to field-measured

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water contents [Sisson, 1987; Sisson et aL, 1988]. The $dK/d\theta$ *values* required in the fitting process can be estimated from implicit solutions of the unit gradient equation:

$$K'(\theta) = \frac{dK}{d\theta} = \frac{z}{t} \tag{1}$$

where $dK/d\theta$ *is* the slope of the $K(\theta)$ function, and z and t are depth and time of the θ observation. Once the parameters of $dK/d\theta$ are available, the $K(\theta)$ function can be calculated immediately. Previous shortcomings to using fitted $dK/d\theta$ functions for estimating hydraulic properties were that the $K(\theta)$ functions had to be of relatively simple form (i.e., exponential or power functions), and that measured h values could not be used in the fitting process [Sisson, 1987]. The primary purpose of this paper is to explore substituting *K'* data (as estimated by Eq. 1) in lieu of the more difficult to obtain *K* data, and to carry out the analysis in terms of a parameter optimization process which permits one to consider all available data. Additional details of the method are given in a recent paper by Sirson and van Genuchten (1991].

THEORY

The soil water retention curve is assumed to be *given* by *[van Genuchten, 1980]*

$$\theta(h) = \theta_r + \frac{\theta_s - \theta_r}{(1+|\alpha h|^n)^m} \qquad (\theta_r \le \theta \le \theta_s) \tag{2}$$

where θ_r and θ_s are the residual and saturated water contents, respectively; and where a, m, and n are parameters estimated by curve fitting. This paper will consider only the restricted case where m is given by $m=$ l-l/n. The unsaturated hydraulic conductivity function was obtained by combining (2) with the pore-size distribution model of *Mualem* [1976]

$$K(S_e) = K_s\sqrt{S_e}\left[\frac{\int_0^{S_e}\frac{dx}{h(x)}}{\int_0^{1}\frac{dx}{h(x)}}\right]^2 \tag{3a}$$

Combining (2) with (3a). and letting m= l-l/n, gives

$$K(S_e) = K_s\sqrt{S_e}\left[1-(1-S_e^{1/m})^m\right]^2 \tag{3b}$$

or

$$K(h) = K_s\frac{[(1+|ah|^n)^m-|ah|^{n-1}]^2}{(1+|\alpha h|^n)^{5m/2}} \tag{3c}$$

where

$$S_e = \frac{\theta - \theta_r}{\theta_s - \theta_r} \tag{4}$$

The K' function required for the unit gradient (or gravity-drainage) model is

$$K'(\theta) = K_s \frac{(1 - A^m)(1 - A^m + 4S^{1/m} A^{m-1})}{2(\theta_s - \theta_r)\sqrt{S_e}} \tag{5}$$

where

$$A = 1 - S_e^{1/m} \tag{6}$$

Note that the hydraulic functions above contain five potentially unknown parameters, represented here by the five-parameter vector $\mathbf{b} = \{K_s, \theta_r, \theta_s, \mathrm{a}, \mathrm{n}\}$.

The computer program RETC [van Genuchten et al., 1991] estimates **b** from observed data using a nonlinear curve fitting procedure based on Marquardt's maximum neighborhood method [Marquardt, 1963]. The objective function to be minimized in RETC is of the form

$$O(\mathbf{b}) = \sum_{i=1}^{M} \left\{ w_i [\theta_i^* - \theta_i(\mathbf{b})] \right\}^2 + \sum_{i=M+1}^{N} \left\{ W_1 W_2 w_i [\ln(K_i^*) - \ln(K_i(\mathbf{b})] \right\} \tag{7}$$

where θ_i^* and K_i^* are the measured water contents and hydraulic conductivities. respectively; while $\theta_i(\mathbf{b})$ and $K_i(\mathbf{b})$ are the computed values for each successive estimate of **b** by using (2) and (3). respectively. Three sets of weights are used in the objective function: w_i, W_1, and W_2. *The* weights w_i may be used to weigh each individual data point individually. W_1 allows weighing of the retention data relative to the hydraulic conductivity data, and W_2 is calculated internally in RETC using

$$W_2 = \frac{\frac{1}{M} \sum_{i=1}^{M} w_i \theta_i^*}{\frac{1}{N-M} \sum_{i=M+1}^{N} w_i \,|\ln(K_i^*)|} \tag{8}$$

which gives the water content data approximately the same weight as the In *K* data in the fitting process. The w_i weights in this study were set to 1 and, in most cases, W_1 to 0.1.

The procedure for estimating the unknown parameter vector **b** directly from unit-gradient data is exactly the same as for the *K-θ-h* data, except that *K* in (7) is replaced by K'(B). the observed values of which are given immediately by z/t at given measured values of $\theta_i^*(z,t)$ as indicated by (1). while the predicted values of K'(0) are given by (5).

METHODS

The unit-gradient based optimization approach was applied to four data sets taken from the Literature. The data sets were for a Norfolk sand (Thermic Typic Paleudult) from South Carolina ***[Quisenberry et al.,*** 1987], a Troup Loamy sand (Grossarenic Paleudult) from Alabama [Dane et ***aL,*** 1983], a Bethany loam (Thermic Pachic Argiustoll) from Oklahoma [Nofziger et aL, 1983], and a Muir silt loam (Mesic Pachic Haplustoll] from Kansas [Sisson et ***aL,*** 1988]. Results of particle size analysis and bulk density determinations are given in Table 1.

Field water contents for the Norfolk sand were estimated from tensiometers and laboratory water retention curves using small undisturbed soil cores. After flooding the 3 by 3 m square field plot until no significant changes in tensiometer readings were observed, the plot was covered with plastic and allowed to drain freely. All field data were obtained during the drainage phase at the 15.2-cm depth for run "1" at site "1". Complete experimental details are given by ***Quisenberry et al.*** [1987]. The hydraulic functions were fitted to data listed in Tables N1.2, N1.4 and N1.5 of ***Quisenberry et aL*** [1987]. These tables give, respectively, laboratory-measured soil water retention data, field water contents during the drainage phase, and hydraulic conductivity data computed with the standard instantaneous profile method.

In contrast to the Norfolk soil, the Troup, Bethany and Muir soils were instrumented to simultaneously obtain water content and pressure head data. Thus, no laboratory data were used in curve fitting. Field water contents were estimated using neutron probes and pressure heads using tensiometer readings. Data analyzed here were taken from Tables 1.3.3 and 1.51 of Dune et ***aL,*** 1983] for the Troup Ap horizon, Tables 7.1 and 9.1 of Nofziger et ***aL*** [1983] for the Bethany 0-15-cm depth, and from figures in Sisson et aL [1988] for the 140-cm depth horizon of the Muir soil.

The following five combinations of data sets were used in the parameter estimation process:

SET 1: Measured $\theta(h)$ data, together with one hydraulic conductivity value to match the K function to the K data as suggested by Jackson et aL [1965]; the matching K was taken at a conductivity value somewhat less than saturation as recommended by Luckner et al. [1989],

SET 2: Measured $\theta(h)$ data simultaneously with measured $K(\theta)$ data,

SET 3: Measured $K(\theta)$ data, together with one $\theta(h)$ data point to estimate a in (2);

SET 4: Measured $\theta(h)$ data simultaneously with measured $K'(\theta)$ data as estimated with (I), and

SET 5: Measured $K'(\theta)$ data, together with one $\theta(h)$ data point to estimate α in (2).

TABLE 1. Particle Size Fractions and Bulk Densities for the Norfolk Sand, Troup Loamy Sand, Bethany Loam, and Muir Silt Loam Soils

	Sand g/g	silt g/g	Clay g/g	Bulk Density Mg/m^3
Norfolk sand	0.78	0.18	0.04	1.79
Troup loamy sand	0.84	0.13	0.03	1.64
Bethany loam	032	0.46	0.22	1.54
Muir silt loam	0.18	0.54	0.28	1.40

RESULTS AND DISCUSSION

Table 2 lists the fitted parameter values using the five data sets described in the Methods section.. Calculated retention (Eq. 2) and conductivity (Eq. 36) curves using these parameter values are given in Figures 1, 2, and 3 for the Norfolk, Troup and Bethany soils, respectively. Also included in these figures are the observed soil water retention and hydraulic conductivity values as reported in the original studies by *Quisenberry et aL* [1987], ***Dane*** *et aL* [1983] and *Nofziger et aL* [1983].

Figure 1 shows that the observed water retention and hydraulic conductivity data of the Norfolk soil are described well by curves obtained with data sets 2 and 4, both of which included observed retention data in the parameter estimation process (the curves are indicated by the open and solid triangles, respectively). This indicates that K values can be reliably substituted for **K values** when estimating the unsaturated hydraulic properties. Results for data sets 3 and 5, which included only a matching point for the retention curve, are shown in Figure 1 by small open and solid circles, respectively.

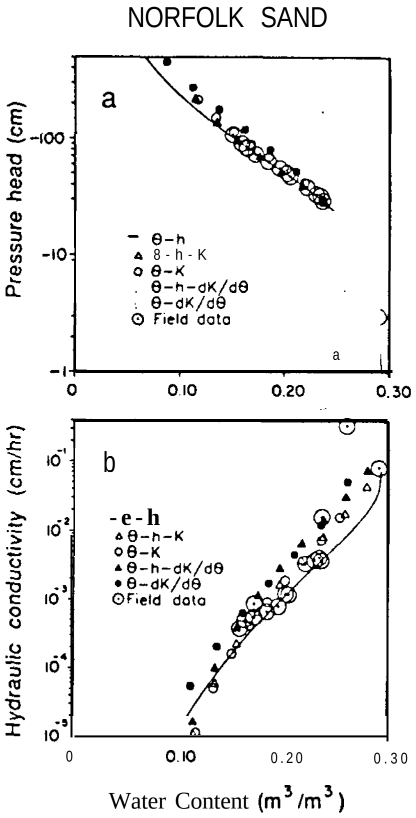

Fig. 1. Soil water retention and hydraulic conductivity curves for Norfolk sand. Open symbols denote curves fitted to K data, closed symbols denote fitted curves using K' data [after Sisson *and van Genuchten,* 1991].

TABLE 2. Fitted Hydraulic Parameters in Equations (2) and (3) for Norfolk Sand, Troup Loamy Sand, Bethany Loam, and Muir Silt Loam

	SET 1	SET 2	SET 3	SET 4	SET 5
			Norfolk *sand*		
8, (m^{3}m^{-3})	0.000	0.064	0.074	0.065	0.000
8, (m'm")	0.293	0.287	0.260	0.2%	0.263
α (an")	0.0307	0.0302	0.0193	0.0312	0.0179
n	1.54	1.78	2.03	1.78	152
K_s (cm/d)	0.072	0.100	0.029	0.190	0.101
r^2	0.998	0.82	0.83	0.97	0.95
			Troup loamy sand		
θ_r (m^{3}m^{-3})	0.000	0.094	0.092	0.083	0.056
θ_s (m^{3}m^{-3})	0.304	0323	0.304†	0.284	0.312
α (cm^{-1})	0.0208	0.0320	0.0250	0.0263	0.0321
n	2.15	2.33	2.18	2.14	1.59
K_s (cm/d)	6.32	10.6	8.57	4.06	24.2
r^2	0.990	0.986	0.975	0.995	0.993
			Bethany loam		
8, (m'm")	0.192	0.202	0.000	0.205	0.000
θ_s (m'm")	0.430	0.427	0.433	0.379	0.404
α (cm^{-1})	0.0206	0.0199	0.530	0.0110	0.0224
n	1.73	1.80	1.167	2.15	1.21
K_s (cm/d)	0.0790	0.0729	2.15	0.0192	0.504
r^2	0.998	>0.999	>0.999	0.93	0.90
			Muir silt loam		
θ_r (m^{3}m^{-3})	0.274	0.229	0.109	0.263	0.260
θ_s (m'm")	0.374	0.361	0.361	0.363	0.363
α (cm^{-1})	0.013	0.009	0.009	0.011	0.014
n	2.51	2.04	1.35	2.31	1.88
K_s cm/d)	15.6	4.60	10.9	15.4	22.9
r^2	0.991	0.892	0.794	0.972	0.963

† Fixed at 0.304 (SET I) because of perfect correlation between θ_s and K_s.

These two data sets produced relatively low estimates of θ_s, in part because of a lack of measured field data close to saturation. This suggests that retention data near saturation are important for producing reliable estimates of the hydraulic properties. The very similar curves produced by data sets 3 and 5, and the nearly identical curves resulting from data sets 2 and 4. show that $dK/d\theta$ and K data are equally effective in the parameter estimation process. Figure ***lb*** also gives the predicted hydraulic conductivity curve using hydraulic parameters calculated from the observed retention data (data set 1). Relatively good agreement was obtained with the observed conductivity data, except for one relatively high value near saturation (Fig. lb).

Results for the Troup loamy sand (Fig. 2) are similar to those of the Norfolk sand in that the fitted retention curves coalesce nicely over the range of water contents where data are available. The parameter θ_s was found to be perfectly correlated to K_s for data set 3. For this reason, θ_s was fixed at 0.304, being the value estimated from data set 1. The fitted retention curves deviate from each other primarily at the lower water contents, largely because of differences in the estimated θ_r values (Fig. 2a). Since the Troup field data used in our analysis were relatively far removed from the residual water content, the estimated θ_r values should be viewed as extrapolated values subject to considerable error and without much physical meaning.

The hydraulic conductivity curves obtained with data sets 2 through 5 (Fig. 2b) for the Troup soil are essentially identical and agree closely with the field-measured hydraulic conductivities. However, the solid $K(\theta)$ curve calculated from retention data

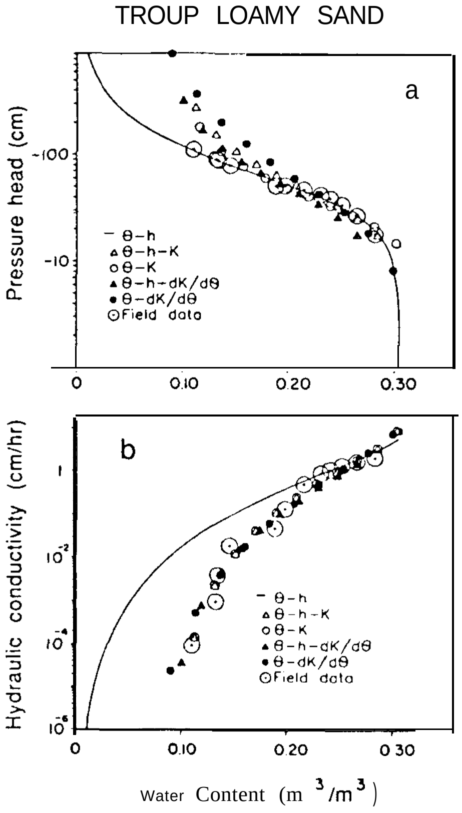

Fig. 2. Soil water retention and hydraulic conductivity curves for Troup loamy sand. Open symbols denote curves fitted to K data, closed symbols denote fitted curves using K' data [after Sisson *and van Genuchten,* 1991].

only (plus a single matching point to determine K_s) deviates significantly from the observed conductivity data in the dry range. Hence, we conclude that, at least for the Troup soil, observed conductivity data $K(\theta)$, or gravity-drainage data $K'(\theta)$, must be used in the fitting process to produce reliable estimates of the conductivity function.

Figure 3 shows the experimental and fitted curves for the Bethany soil. Similar trends occur as for the Norfolk and Troup soils in that the curves tend to slightly deviate from each other at relatively low and high water contents, primarily because of a lack of measured retention data in those regions. However, all five data sets produced very similar curves; the overall agreement with the measured data is also good to excellent. Also, notice that the predicted conductivity curve derived from the retention data (data set 1) in this case produced an excellent fit with the observed conductivity data.

Results for the Muir silt loam were found to qualitatively the same as those in Figure 3 for the Bethany soil, and are nor further show here. The five data sets produced essentially overlapping curves over the range of the observed data, although some major differences occurred in the extrapolated values of θ_r and θ_s. Thus, if accurate estimates of the extreme dry and wet ends of the retention curve are required, then data must be obtained in these regions to ensure correct estimates of $\theta(h)$ or K(h).

CONCLUSIONS

Results for the four soils considered here show that $K'(\theta)$ data can be reliably substituted for $K(\theta)$ data in fitting retention and hydraulic conductivity curves. This finding is significant for field experimentaiists since $K'(\theta)$ is quite easily obtained during drainage of an initially saturated field soil. As shown by (1). $K'(\theta)$ is simply equal to the ratio z/t when $\theta(z,t)$ is measured. By comparison, $K(\theta)$ data require a much more elaborate and time-consuming sampling program to ensure meaningful data after interpolating and differencing field data.

Our study indicates two extensions to the analysis of gravity-drainage experiments. One extension is a result from the fact that the instantaneous profile analysis is now formulated in the form of a parameter optimization process. This approach allows one to extend the range of experimental data by augmenting the database with measurements that are obtained independently from the gravity-drainage experiment. For example, when estimates of the pressure head at relatively low water contents are available from laboratory- or field-measured soil water depletion experiments, or perhaps from psychrometer studies, then those estimates can be immediately incorporated in the optimization analysis. Another extension results when K' data are substituted for K data. Since K'(0) can be estimated with great precision, the accuracy of the estimation process should also improve.

Of more subtle importance to the field investigator is the possibility to redesign drainage experiments to better control the noise and range of observed water contents, pressure heads and hydraulic conductivities that will enter the curve fitting process. For example, since K and K' often generate approximately linear curves when plotted using a logarithmic scale, a better distribution of data points may be obtained when K and K are measured at time values of, for example, 1/8, 1/4, 1/2, 1, ... and 128 days. This is easily accomplished for the K' variable since its value can be computed prior to sampling θ or h. By comparison, K is not known until after the data are analyzed. Long time intervals between readingscauses serious interpolation problems when the standard

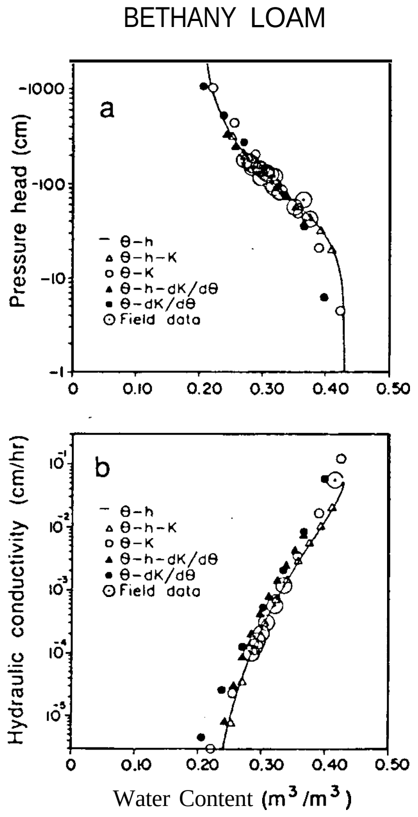

Fig. 3. Soil water retention and hydraulic conductivity curves for Bethany loam. Open symbols denote curves fitted to *K* data, closed symbols denote fitted curves using K' data (after Sisson *and van Genuchten,* 1991].

instantaneous profile method is used, but would cause no problems for the K' method. Hence, the gravity-drainage method using K' can also be more cost-effective by eliminating several trips to the field.

The differences in fitted parameters from the five data sets used in our study could be reduced if special efforts had been spent to obtaining more precise estimates of θ_r, θ_s and K_s. For example, determining water contents of soil samples obtained while instrumenting the site under dry conditions would further improve the accuracy of the θ_r estimates. Similarly, the precision of θ_s could be improved by averaging several water content values obtained immediately before initiation of the drainage phase. Also, the shape of the $K'(\theta)$ function near saturation would be more precisely known if additional water content data from shallow depths were available during the first few minutes of the gravity-drainage phase. With the shape of the $K'(\theta)$ close to saturation more precisely known, the precision of the fitted K_s would then also improve.

REFERENCES

Ahuja, L. R., R. E. Green, S. K. Chong, and D. R. Nielsen. 1980. A simplified functions approach for determining soil hydraulic conductivities and water characteristics in situ. ***Water Resow. Res. 16:947-953.***

Dane, J. H., D. K. Cassel, J. M. Davidson, W. L. Pollans, and V. L. Quisenbeny. 1983. Physical characteristics of soils of the Southern Region - Troup and Lakeland Series. South. ***Coop. Ser. Bull. 262,*** Alabama Agric. Exp. Sta., Auburn Univ., Auburn, AL.

Fluhler, H., M. S. Ardakani, and L. H. Stolzy. 1976. Error propagation in determining hydraulic conductivities from successive water content and pressure head profiles. ***Soil Sci. Soc. Am J.*** 40:830-836.

Jackson, R. D., R. J. Reginato, and C. H. M. Van Bavel. 1965. Comparison of measured and calculated hydraulic conductivities of unsaturated soils. ***Water Resow. Res. 1:375-380.***

Libardi, P. L., K. Reichardt, D. R. Nielsen, and J. W. Biggar. 1980. Simple field methods for estimating soil hydraulic conductivity. ***Soil Sci Soc. Am*** *J.* 44:3-7.

Luckner, L., M. Th. van Genuchten and, D. R. Nielsen. 1989. A consistent set of parametric models for the two-phase flow of immiscible fluids in the subsurface. ***Water Resour. Res. 25:2187-2193.***

Luxmoore, R. J., T. Grizzard, and M. R. Patterson. 1981. Hydraulic properties of Fullerton cherty silt loam. ***Soil Sci Soc. Am J.*** 45:692-698.

Marquardt, D. W. 1963. An algorithm for least-squares estimation of nonlinear parameters. J. Soc. Ind ***Appl. Math. 11:431-441.***

Mualem, Y. 1976. A new model for predicting the hydraulic conductivity of unsaturated porous media. ***Water Resow. Res. 12:513-522.***

Nofziger, D. L., J. L. Williams, A. G. Hornsby, and A. L. Wood. 1983. Physical characteristics of soils of the Southern Region - Bethany, Konawa, and Tipton Series. ***South. Coop. Ser.*** *Bull.* 265, Agric. Exp. Sta., Oklahoma State Univ., Stillwater, OK.

Quisenbeny, V. L., D. K Cassel, J. H. Dane, and J. C. Parker. 1987. Physical characteristics of soils of the Southern Region - Norfolk, Dothan, Wagram, and Goldsboro Series. ***South. Coop. Ser. Bull.*** 263, South Carolina Agric. Exp. Stat., Clemson University, Clemson, SC.

Rose, C. W., W. R. Stern, and J. E. Drummond. 1965. Determination of hydraulic conductivity as a function of depth and water content for soil in situ. ***Water*** Resour. ***Res.*** 3:1-9.

Sisson, J. B. 1987. Drainage from layered field soils: Fixed gradient models. ***Water Resour. Res. 23:2071-2075.***

Sisson, J. B., W. M. Klittich, and S. B. Salem. 1988. Comparison of two methods for summarizing hydraulic conductivities of a layered soil. ***Water Resour. Res. 24:1271-1276.***

Sisson, J. B., and M. Th. van Genuchten. 1991. An improved analysis of gravity drainage experiments for estimating the unsaturated hydraulic functions. ***Water Resour. Res. 27:569-575.***

van Genuchten, M. Th. 1980. A closed-form equation for predicting the hydraulic conductivity of unsaturated soils. ***Soil Sci Soc. Am J.*** 44:892-898.

van Genuchten, M. Th., F. J. Leij, and S. R. Yates. 1991. The RETC code for quantifying the hydraulic properties of unsaturated soils. EPA/600/2-91/065. 93 pp. R. S. Kerr Environ. Res. Lab., U. S. Environmental Protection Agency, Ada, OK.